

CS 320  
Computer Language Processing  
Exercises: Weeks 1 and 2

February 28, 2025

## 1 Languages and Automata

**Exercise 1** Consider the following languages defined by regular expressions:

1.  $\{a, ab\}^*$
2.  $\{aa\}^* \cup \{aaa\}^*$
3.  $a^+b^+$

and the following languages defined in set-builder notation:

- A.  $\{w \mid \forall i. 0 \leq i \leq |w| \wedge w_{(i)} = b \implies (i > 0 \wedge w_{(i-1)} = a)\}$
- B.  $\{w \mid \forall i. 0 \leq i < |w| - 1 \implies w_{(i)} = b \implies w_{(i+1)} = a\}$
- C.  $\{w \mid \exists i. 0 < i < |w| \wedge w_{(i)} = b \wedge w_{(i-1)} = a\}$
- D.  $\{w \mid (|w| = 0 \pmod 2 \vee |w| = 0 \pmod 3) \wedge \forall i. 0 \leq i < |w| \implies w_{(i)} = a\}$
- E.  $\{w \mid \forall i. 0 \leq i < |w| - 1 \wedge w_{(i)} = a \implies w_{(i+1)} = b\}$
- F.  $\{w \mid \exists i. 0 < i < |w| - 1 \wedge (\forall y. 0 \leq y \leq i \implies w_{(y)} = a) \wedge (\forall y. i < y < |w| \implies w_{(y)} = b)\}$

For each pair (e.g. 1-A), check whether the two languages are equal, providing a proof if they are, and a counterexample word that is in one but not the other if unequal.

**Solution** Equal language pairs:  $1 \mapsto A, 2 \mapsto D, 3 \mapsto F$ .

Counterexamples ( $\cdot^*$  means the word is in the alphabet-labelled language, and the number-labelled language otherwise):

	A	B	C	D	E	F
1	-	a	a	a	aa	a
2	ab*	ba*	ab*	-	ab*	aa
3	abb	abb	aba*	aaabb	aab	-

We prove the first case as an example.

$$\{a, ab\}^* = \{w \mid \forall i. 0 \leq i \leq |w| \wedge w_{(i)} = b \implies (i > 0 \wedge w_{(i-1)} = a)\}$$

We must prove both directions, i.e. that  $\{a, ab\}^* \subseteq \{w \mid P(w)\}$  and that  $\{w \mid P(w)\} \subseteq \{a, ab\}^*$ .

**Forward:**  $\{a, ab\}^* \subseteq \{w \mid P(w)\}$ :

We must show that for all  $w \in \{a, ab\}^*$ ,  $P(w)$  holds. For any  $i \in \mathbb{N}$ , given that  $0 \leq i \leq |w| \wedge w_{(i)} = b$ , we need to show that  $i > 0 \wedge w_{(i-1)} = a$ .

From the definition of  $*$  on sets of words, we know that there must exist  $n < |w|$  words  $w_1, \dots, w_n \in \{a, ab\}$  such that  $w = w_1 \dots w_n$ . The index  $i$  must be in the range of one of these words, i.e. there exist  $1 \leq m \leq n$  and  $0 \leq j < |w_m|$  such that  $w_{(i)} = w_{m(j)}$ .

We know that  $w_{(i)} = b$  and  $w_m \in \{a, ab\}$  by assumption. The case  $w_m = a$  is a contradiction, since it cannot contain  $b$ . Thus,  $w_m = ab$ . We know that  $w_{(i)} = w_{m(j)} = b$ , so  $j = 1$ . Thus,  $w_{(i-1)} = w_{m(j-1)} = w_{m(0)} = a$ , as required. Since  $i - 1 \geq 0$ , being an index into  $w$ ,  $i > 0$  holds as well. Hence,  $P(w)$  holds.

**Backward:**  $\{w \mid P(w)\} \subseteq \{a, ab\}^*$ :

We must show that for all  $w$  such that  $P(w)$  holds,  $w \in \{a, ab\}^*$ . We know by definition of  $*$  again, that  $w \in \{a, ab\}^*$  if and only if there exist  $n < |w|$  words  $w_1, \dots, w_n \in \{a, ab\}$  such that  $w = w_1 \dots w_n$ . We attempt to show that if  $P(w)$  holds, then  $w$  admits such a decomposition.

We proceed by induction on the length of  $w$ .

*Induction Case*  $|w| = 0$ : The empty word has a decomposition  $w = \epsilon$  (with  $n = 0$  in the decomposition). QED.

*Induction Case*  $|w| = 1$ : The word  $w$  is either  $a$  or  $b$ . We know that  $P(w)$  holds, so  $w = a$  (why?). The decomposition is  $w = a$ , with  $n = 1$  and  $w_1 = a$ . QED.

*Induction Case*  $|w| > 1$ :

Induction hypothesis: for all words  $v$  such that  $|v| < |w|$  and  $P(v)$  holds,  $v$  admits a decomposition into words in  $\{a, ab\}$ , and thus  $v \in \{a, ab\}^*$ .

We need to show that if  $P(w)$  holds, then  $w$  admits such a decomposition as well. Split the proof based on the first two characters of  $w$ . There are four possibilities. We give the name  $v$  to the rest of  $w$ .

1.  $w = aav$ :  $P(w)$  holds, so  $\forall i. 0 \leq i \leq |w| \wedge w_{(i)} = b \implies (i > 0 \wedge w_{(i-1)} = a)$ . In particular, we can restrict to  $i > 1$  as

$$\forall i. 2 \leq i \leq |w| \wedge w_{(i)} = b \implies (i > 0 \wedge w_{(i-1)} = a)$$

but  $w_{(i)}$  for  $i \geq 2$  is simply  $v_{(i-2)}$ . Rewriting:

$$\forall i. 2 \leq i \leq |w| \wedge v_{(i-2)} = b \implies (i > 0 \wedge v_{(i-3)} = a)$$

Finally, since the statement holds for all  $i$ , we can replace  $i$  by  $i+2$  without loss of generality, using  $|v| = |w| - 2$ :

$$\forall i. 0 \leq i \leq |v| \wedge v_{(i)} = b \implies (i > 0 \wedge v_{(i-1)} = a)$$

This is precisely the statement  $P(v)$ , so by the induction hypothesis,  $v$  has a decomposition into words in  $\{a, ab\}$ ,  $v = v_1 \dots v_m$  for some  $m < |v|$  and  $v_i \in \{a, ab\}$ .

We can now construct a decomposition for  $w$ ,  $w = w_1 \dots w_{m+2}$  such that  $w_1 = a$ ,  $w_2 = a$ , and  $w_{i+2} = v_i$  for  $1 \leq i \leq m$ . Since  $m < |v|$  and  $|v| = |w| - 2$ ,  $m + 2 < |w|$ . QED.

2.  $w = abv$ : by the same argument as the previous case,  $v$  has a decomposition into words in  $\{a, ab\}$ ,  $v = v_1 \dots v_m$  for some  $m < |v|$  and  $v_i \in \{a, ab\}$ .

We can similarly construct a decomposition for  $w$ ,  $w = w_1 \dots w_{m+1}$  such that  $w_1 = ab$  and  $w_{i+1} = v_i$  for  $1 \leq i \leq m$ . Since  $m < |v|$  and  $|v| = |w| - 2$ , in particular  $m + 1 < |w|$ . QED.

3.  $w = bav$  or  $w = bbv$ :  $P(w)$  cannot hold (set  $i = 0$ ), so the statement is vacuously true.

□

**Exercise 2** For each the following languages, construct an NFA  $\mathcal{A}$  that recognizes them, i.e.  $L(\mathcal{A}) = L_i$ :

1.  $L_1$ : binary strings divisible by 3
2.  $L_2$ : binary strings divisible by 4
3.  $L_3$ :  $\{(w_1 \oplus w_2) \mid w_1 \in L_1 \wedge w_2 \in L_2 \wedge |w_1| = |w_2|\}$

where  $\oplus$  is the bitwise-xor operation on binary strings.

### Solution

1. The language of binary strings divisible by 3. We need two observations to construct this automaton:
  - (a) If the automaton has consumed a binary string  $s$  with decimal value, say,  $val(s) = n$ , then we can determine the decimal value of the string after reading one more character as either  $val(s0) = 2n$  or  $val(s1) = 2n + 1$ .
  - (b) The set of strings is finite, but it is sufficient to know only the value of the string *modulo 3* to determine if it is divisible.

We construct the automaton  $\mathcal{A}_1 = (Q, \Sigma, \delta, q_{init}, F)$  where:

- $Q = \{q_{init}, q_0, q_1, q_2\}$ , representing the initial state (empty word has no value), and the states corresponding to the values 0, 1, 2 modulo 3.
- $\Sigma = \{0, 1\}$ , as required.
- $\delta = \{(q_i, 0, q_j) \mid 2i \bmod 3 = j\} \cup \{(q_i, 1, q_j) \mid (2i + 1) \bmod 3 = j\} \cup \{(q_{init}, 0, q_0), (q_{init}, 1, q_1)\}$ , i.e., there is a transition from  $q_i$  to  $q_j$  if, as the currently known value modulo 3 is  $i$ , on reading 0 the next value is  $j = 2i \bmod 3$ . We use the fact that  $2n \bmod 3 = 2(n \bmod 3) \bmod 3$ . The case for reading 1 is similar. The translations from the

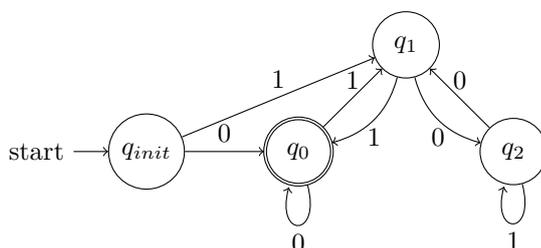
initial state are to the states corresponding to the values 0, 1 modulo 3.

For example, if we have read “1101”, with decimal value 13, we must be in state  $q_1$ , as  $13 \bmod 3 = 1$ . On reading a 0, we have the string “11010” with decimal value 26, and  $26 \bmod 3 = 2$ , so we transition to  $q_2$ .

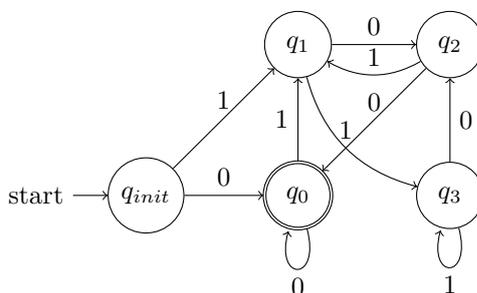
The full automaton is below.

- $F = \{q_0\}$  as we accept that words that are divisible by 3, and are hence equal to 0 modulo 3.

The automaton is:



- The language of binary strings divisible by 4. The construction is similar to the one above, now with 5 states.



- To compute the bitwise-xor of two strings, we must compute a product automaton. To accept a word  $w$ , there must exist  $w_1, w_2$  such that  $w_1 \in L_1$ , and  $w_2 \in L_2$ .

We do not explicitly construct the automaton, but present an argument. First, consider the truth table for xor:

$b_1$	$b_2$	$b_1 \oplus b_2$
0	0	0
0	1	1
1	0	1
1	1	0

Notably, given a xor result, we cannot exactly determine the input bits. In essence, we construct an automaton that, given a string, tries to simulate the two input automata in parallel non-deterministically on all possible

pairs of input strings. If any of them are accepted, that means we found a pair of strings that, one, are accepted by the two original automata, and two, have the input string as their bitwise-xor.

Formally, the automaton  $\mathcal{A}_3 = (Q, \Sigma, \delta, q_{init}, F)$  has:

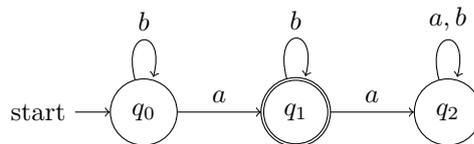
- $Q = Q_1 \times Q_2$ , where  $Q_1$  and  $Q_2$  are the state sets of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .
- $\Sigma = \{0, 1\}$  as before.
- $q_{init} = (q_{1,init}, q_{2,init})$  where  $q_{1,init}$  and  $q_{2,init}$  are the initial states of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .
- $F = F_1 \times F_2$  similarly.
- $\delta$  is constructed as follows: for a pair of states  $(q_1, q_2)$ , on reading a 0, we look at the truth table of xor; two input pairs  $(0, 0)$  and  $(1, 1)$  could have produced this result bit. Hence, we add transitions for both automata simultaneously,  $((q_1, q_2), 0, (q'_1, q'_2))$  corresponding to possible inputs  $(0, 0)$  if  $\delta_1(q_1, 0, q'_1)$  and  $\delta_2(q_2, 0, q'_2)$ , and similarly  $((q_1, q_2), 0, (q'_1, q'_2))$  corresponding to possible inputs  $(1, 1)$  if  $\delta_1(q_1, 1, q'_1)$  and  $\delta_2(q_2, 1, q'_2)$ .

The case for reading a 1 is similar, with possible input pairs  $(0, 1)$  and  $(1, 0)$ .

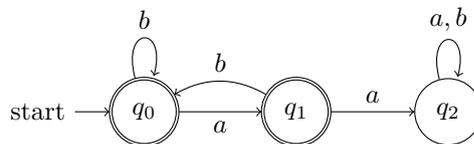
□

**Exercise 3** Give a verbal and a set-notational description of the language accepted by each of the following automata. You can assume that the alphabet is  $\Sigma = \{a, b\}$ .

1.  $\mathcal{A}_1$



2.  $\mathcal{A}_2$



**Solution**

- As regular expression:  $b^*ab^*$ , this is the language of words that contain exactly one  $a$ . In set-notation:

$$\{w \mid \exists! i. 0 \leq i < |w| \wedge w_{(i)} = a\}$$

2. As generalized regular expression (with complement):  $(\Sigma^*aa\Sigma^*)^c$ . Without complement:  $b^*(ab^+)^*(a \mid \epsilon)$ . This is the language of words that contain no consecutive pair of  $a$ 's. In set-notation:

$$\{w \mid \forall i. 0 \leq i < |w| \wedge w_{(i)} = a \implies (i + 1 \geq |w| \vee w_{(i+1)} \neq a)\}$$

□

## 2 Lexing

Consider a simple arithmetic language that allows you to compute one arithmetic expression, construct conditionals, and let-bind expressions. An example program is:

```
let x = 3 in
let y = ite (x > 0) (x * x) 0 in
  (2 * x) + y
```

The lexer for this language must recognize the following tokens:

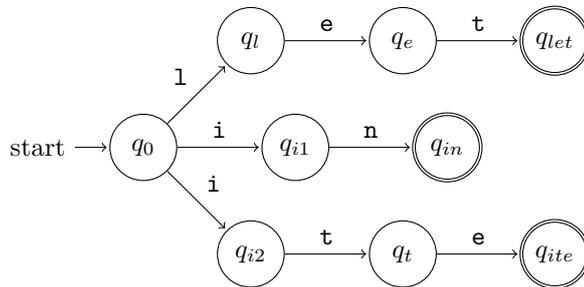
```
keyword : let | in | ite
         op : + | - | * | /
         comp : > | < | == | <= | >=
         equal : =
         lparen : (
         rparen : )
         id : letter · (letter | digit)*
         number : digit+
         skip : whitespace
```

For simplicity, *letter* is a shorthand for the set of all English lowercase letters  $\{a - z\}$  and *digit* is a shorthand for the set of all decimal digits  $\{0 - 9\}$ .

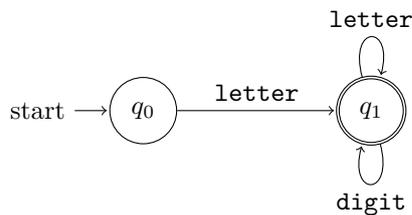
**Exercise 4** For each of the tokens above, construct an NFA that recognizes strings matching its regular expression.

**Solution** The construction is similar in each case, following translation of regular expressions to automata. For example:

- keyword: `let | in | ite`



- `id: letter · (letter | digit)*`



The other cases are similar. □

A lexer is constructed by combining the NFAs for each of the tokens in parallel, assuming maximum munch. The resulting token is the first NFA in the token order that accepts a prefix of the string. Thus, tokens listed first have higher priority. We then continue lexing the remaining string. You may assume that the lexer drops any `skip` tokens.

**Exercise 5** For each of the following strings, write down the sequence of tokens that would be produced by the constructed lexer, if it succeeds.

1. `let x = 5 in x + 3`
2. `let5x2`
3. `xin`
4. `==>`
5. `<==><==`

**Solution**

1. `[keyword("let"), id("x"), equal("="), number("5"), keyword("in"), id("x"), op("+"), number("3")]`
2. `[keyword("let"), number("5"), id("x2")]`
3. `[id("xin")]`
4. `[comp("=="), comp(">")]`
5. `[comp("<="), comp("=="), comp(">"), comp("<="), equal("=")]`

□

**Exercise 6** Construct a string that would be lexed differently if we ran the NFAs in parallel and instead of using token priority, simply picked the longest match.

**Solution** There are many possible solutions. The key is to notice which tokens have overlapping prefixes.

An example is `letx1`, which would be lexed as `[keyword("let"), id("x1")]` if we check acceptance in order of priority, but as `[id("letx1")]` if we run them in parallel. □